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On the generation and radiation of magneto-acoustic waves

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The generation, and thence the dissipation, propagation and radiation, of waves in a compressible fluid subjected to a magnetic field is studied as an extension of the 'acoustic analogy' (Lighthill 1952) to magneto-acoustics. A formal theory of magneto-acoustic waves introduces (i) a single differential operator describing propagation, (ii) a dynamic and a magnetic tensor modelling generation and (iii) a dissipation tensor to complete the wave equation. The interpretation of these tensors indicates the magnitude of the physical processes of wave generation, by turbulence and inhomogeneities, and of wave dissipation, by viscous and electrical resistance and heat conduction. The total quadrupole components are classified according to the mode of emission. If the magnetic field or compressibility is neglected we obtain, respectively, 'aerodynamic acoustics' and a corresponding theory for Alfvén waves. These hydrodynamic and hydromagnetic results contrast with magnetodynamics, when the magnetic field is dominant. The magneto-acoustic far field implies a law of directivity and intensity of radiation. The main results have been collected in three summary tables (see appendix).

1. Introduction

This paper is intended as an extension of recent research both on aerodynamic acoustics and on hydromagnetics. The subject implied is the study of waves in a free flow of a compressible fluid subjected to an external magnetic field. The main topics are generation, propagation, dissipation and radiation of magneto-acoustic waves.

1.1. *Sound generation by flows*

The scientific study of the propagation of sound is almost as old as mathematics itself, the two having followed a generally parallel development from the pioneers in classical Greece (e.g. Pythagoras 6th century B.C.; see Jeans 1968), through the founders of mathematical physics (e.g. D'Alembert 1747) to the treatises of the nineteenth century (Rayleigh 1877). The generation of sound by vibrating bodies (e.g. strings and membranes) and fluctuating mass sources (e.g. sirens and pipes) was studied extensively, these problems having the distinct property that the sources and the medium of propagation are physically separate.

An area left largely unaddressed consisted of cases where the medium of propagation is itself the source, such as the aerodynamic noise produced in a flow by its internal forcing mechanisms. This problem was originally considered by Lighthill (1952), who used a conceptual 'analogy' to 'separate' source from medium. Thence it was concluded that, from the point of view of generation of sound, turbulence can be modelled by a quadrupole, known as the Lighthill tensor. The main prediction of the 'acoustic analogy', namely the dependence of the intensity of radiation on the *eighth* power of the turbulent velocity, has been basically confirmed by experiment.

This first success of the theory of aerodynamic acoustics has prompted various attempts at extension, to include the effects of solid boundaries (Curle 1955), boundary layers (Phillips 1960), convection of the source (Ffowcs Williams 1963) and moving surfaces (Ffowcs Williams & Hawkins 1968). The other mechanism of generation of sound naturally present in a flow is the convection of physical inhomogeneities (e.g. of density or temperature) in non-uniform flow. Their consideration for flow in ducts (Candel 1972) and past arbitrary solid bodies (Howe 1975*a*) has suggested an alternative form of the 'analogy' (Howe 1975*b*) which also renders explicit the effect of vorticity (Powell 1961).

1.2. *Fluids in a magnetic field*

The above developments were undertaken within the context of acoustics, taken as a branch of classical fluid mechanics. The latter describes accurately the properties of fluids in our closer environment, such as the water in oceans and basins, the air in the *lower* atmosphere and flows through various machines. However, outside a band of some tens of miles about the surface of the earth, natural fluids are ionized and subjected to magnetic fields whose effects can hardly be neglected.

In the mantle of the earth a complex interaction of dynamic and magnetic-fluid effects (Moffatt 1976) creates an overall magnetic field that is stable. This is significant up to a distance of several radii, capturing part of the solar wind in the magnetosphere (Cowling 1957), which surrounds the earth with ionized gas. The sun, as well as other stars, is basically an ionized fluid held together by its gravitational attraction; it also has a permanent magnetic field, which may contribute to the explanation of such intriguing properties of the chromosphere as the rise in temperature with altitude (Lighthill 1967).

The solar magnetic field could determine the migration of the bright spots observable from the earth (Alfvén 1943); in a possibly similar manner stellar magnetic fields affect the formation and distribution of galactic gas clouds. On an earthly scale, various energy production schemes using plasmas have been proposed (e.g. Shermann & Sutton 1965). The most promising, though difficult, is an 'adaptation' of the energy production process in the sun, namely controlled nuclear fusion.†

1.3. *Magneto-acoustic waves*

Compared with this vast background, knowledge on magneto-acoustic waves is rather limited, yet each new piece of information impresses scientific curiosity as subject worthy of further inquiry. Waves in an *incompressible* fluid subjected to a magnetic field were predicted theoretically by Alfvén (1942), and subsequently observed experimentally (Lundquist 1949). The study of the propagation of magnetic waves was then extended to compressible fluids (Astrom 1950; Herlofson 1950) and to include gravity (Howe 1969), the combined effect of compressibility and gravity being considered by Moore & Spiegel (1964). The application of external force fields to an homogeneous compressible fluid causes waves to become anisotropic, with radiation 'beaming' through conical pencils (Lighthill 1960).

† See *Proc. 2nd UN Conf. on Peaceful Uses of Atomic Energy*, 1958, *Geneva*, vol. 31, *Fusion Devices*.

The study of fluids in an external magnetic field has expanded mainly through its foundation on kinetic theory (Chapman & Cowling 1952), through the study of the stability of flows (Chandrasekhar 1961) and through various cosmological applications (Alfvén & Falthammar 1962). However, at least one fundamental question, perhaps the first to suggest itself if we know of the existence of waves, appears to be still unanswered: how are magneto-acoustic waves generated? To address this question effectively requires consideration of propagation and dissipation and could lead to predictions on the radiation field.

The purpose of the formal theory (§ 2) is to establish a complete wave equation from the *exact* magnetohydrodynamic (MHD) perturbation system (§ 2.1); the latter is first eliminated to form the propagation operator (§ 2.2) and then reconsidered to identify dynamic and magnetic quadrupoles (§ 2.3). The aim of the subsequent inquiry into the underlying physical processes (§ 3) is (i) to model the sources of waves (§ 3.1) and thus find their origin and magnitude, (ii) to assess the causes of dissipation (§ 3.2) and their effect on generation, and (iii) to classify the quadrupole components (§ 3.3) according to modes of emission.

The topics addressed in the concluding application to radiation fields (§ 4) are: (i) consistency with aerodynamic acoustics (§ 4.1) if the magnetic field is neglected and with the corresponding theory for Alfvén waves if, instead, compressibility is ignored; (ii) the modes and sources in the hydrodynamic and hydromagnetic limits (§ 4.2), in contrast with the magnetodynamic case of strong magnetic field; (iii) the wave far field (§ 4.3) and the directivity and intensity of magneto-acoustic radiation. For ease of reference the main results have been collected in three summary tables (see appendix), concerned respectively with propagation, generation and radiation.

2. The complete wave equation

The formal development of this paper is based on the general equations of a fluid in a magnetic field, reviewed *exactly* for perturbed flow. The linear, non-dissipative part leads, by elimination, to a magneto-acoustic operator describing propagation. The remaining nonlinear or dissipative terms form tensors, of dynamic or magnetic origin, modelling generation and dissipation in a complete wave equation.

2.1. *Exact perturbation flow*

The electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ in an isotropic medium (e.g. a fluid of magnetic permeability μ) are specified by Maxwell's equations, of which we mention the curl pair

$$\nabla \wedge \mathbf{E} = -\frac{\mu}{c}\frac{\partial \mathbf{H}}{\partial t} + \nabla \wedge (\mathbf{v} \wedge \mathbf{H}), \quad \nabla \wedge \mathbf{H} = \frac{4\pi}{c}\mathbf{J}. \tag{1}$$

To account for the motion of the fluid with velocity $\mathbf{v}$ we have added to the local time derivative $\partial \mathbf{H}/\partial t$ the convection effect $\nabla \wedge (\mathbf{H} \wedge \mathbf{v})$, forming the total time derivative $D\mathbf{H}/Dt$ (e.g. Landau & Lifshitz 1959, vol. 8, § 49). In the second equation we have neglected the displacement $(4\pi)^{-1}\, D\mathbf{D}/Dt$ compared with the electric current $\mathbf{J}$, which implies that $\nabla . \mathbf{J} = 0$. Thus positive and negative charges must balance to give neutrality, although the fluid may contain electric currents.

For a fluid (or charges) which is not too rarefied, such that statistical (rather than molecular) laws may be used, the current **J** and the electric field **E** are proportional. The coefficient generally depends on the magnetic field **H**, e.g. through a power expansion, with scalar coefficients to assure isotropy:

$$\mathbf{E} = (\sigma)^{-1}\mathbf{J} + \xi^{-1}\,\mathbf{J}\wedge\mathbf{H} + O(H^2). \tag{2}$$

The first term (Ohm's law), in which the electrical conductivity σ is independent of **H**, specifies the current due to collisions, which predominate in high-density plasma. At medium densities charges may spiral (between collisions) in the magnetic field, resulting in a Hall current (the second term).

The current **J** and electric field **E** are specified respectively by (1) and (2) if the magnetic field **H** is known. The latter is found by elimination to satisfy the equation of induction:

$$\partial\mathbf{H}/\partial t + \nabla\wedge(\mathbf{H}\wedge\mathbf{v}) = (c^2/4\pi\mu\sigma)\,\nabla^2\mathbf{H} + (c^2/4\pi\mu\sigma)\,\nabla\wedge\{\mathbf{H}\wedge(\nabla\wedge\mathbf{H})\}. \tag{3}$$

This describes the dissipation $\chi \equiv c^2/4\pi\mu\sigma$, $c^2/4\pi\mu\xi$, of the magnetic field $D\mathbf{H}/Dt$ by the resistivity of the fluid, which consists of the 'Ohmic' component σ^{-1} (e.g. Batchelor 1950) and the 'Hall' component ξ^{-1} (Lighthill 1960). Since the discussions of the Ohmic and Hall dissipative terms are similar, we shall, in the interest of brevity, retain only the former in the equations.

For the description of the flow, it is sufficient to add to (3) the equation of continuity

$$\partial\rho/\partial t + \nabla.\rho\mathbf{v} = 0, \tag{4}$$

which expresses conservation of mass, and the momentum equation, which states the balance between the momentum ρv_i and the hydrodynamic, viscous (τ_{ij}) and magnetic stresses:

$$(\partial/\partial t)\,\rho v_i + (\partial/\partial x_j)\{[\rho v_i v_j + p\delta_{ij}] - \tau_{ij} - (\mu/4\pi)\,[H_i H_j - \tfrac{1}{2}H^2\delta_{ij}]\} = 0. \tag{5}$$

It is noted that both the hydrodynamic and the magnetic stress (respectively the first and second square brackets) consist of anisotropic and isotropic parts, the latter being the hydrodynamic pressure p and the magnetic pressure $\mu H^2/8\pi$.

We assume that the flow results from the superposition on a constant mean state of rest of a non-uniform unsteady perturbation:

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{h}(\mathbf{x},t), \quad \rho = \rho_0 + \rho'(\mathbf{x},t). \tag{6}$$

Besides terms that are linear or nonlinear in the perturbations, we may distinguish those that are non-dissipative or dissipative; the latter are associated with finite conductivity $(\sigma \neq \infty)$, non-zero viscosity $(\tau_{ij} \neq 0)$ and the non-isentropic part of the equation of state $(p = p(\rho, s))$, which appears since $\nabla p - c_0^2\nabla\rho \neq 0$, where $c_0^2 \equiv (\partial p/\partial\rho)_s$ is the speed of sound squared. The condition of non-existence of 'magnetic charges' $\nabla.\mathbf{H} = 0$ [consistent with (3)] leads without approximation to $\nabla.\mathbf{h} = 0$ because the mean state is constant.

The latter is of some slight use in simplifying the general equations (3)–(5) after substitution of the perturbations (6), in which we may now drop the subscript zero from mean-state constants (i.e. $\mathbf{H}_0$, ρ_0 become $\mathbf{H}, \rho$). For subsequent reference we collect the linear, non-dissipative terms on the left-hand sides and retain all other terms (on the right) so that the perturbation equations remain *exact*:

$$\partial h_i/\partial t + H_i\,\partial v_j/\partial x_j - H_j\,\partial v_i/\partial x_j = \nabla \wedge (\mathbf{v} \wedge \mathbf{h}) + (c^2/4\pi\mu\sigma)\,\nabla^2\mathbf{h}, \tag{7a}$$

$$\partial\rho'/\partial t + \rho\,\partial v_j/\partial x_j = -\,\partial(\rho' v_j)/\partial x_j, \tag{7b}$$

$$\partial v_i/\partial t + (c_0^2/\rho)\,\partial\rho/\partial x_i + (\mu/4\pi\rho)\,H_j(\partial h_j/\partial x_i - \partial h_i/\partial x_j) = -\rho^{-1}\,\partial(\rho' v_i)/\partial t$$
$$-\rho^{-1}(\partial/\partial x_j)\,\{\{\rho v_i v_j + (p - c_0^2\rho)\,\delta_{ij} - \tau_{ij}\} - (\mu/4\pi)\,(h_i h_j - \tfrac{1}{2}h^2\delta_{ij})\}. \tag{7c}$$

2.2. *Propagation operator and modes*

To obtain a single equation describing the propagation of linear, non-dissipative magneto-acoustic waves, we could eliminate between the left-hand sides of (7*a–c*) (setting the right-hand sides to zero). If the said equation is to remain non-trivial both in the limit of no magnetic field ($\mathbf{h} = 0$) and in the limit of incompressible fluid ($\rho' = 0$), the velocity perturbation $\mathbf{v}$ should be the wave variable. The elimination can be performed by noting that on application of $\partial/\partial t$ to (7*c*) the other variables appear only in the form $\partial\rho/\partial t$ or $\partial h_i/\partial t$, which can be re-expressed in terms of $\mathbf{v}$ by use of (7*a, b*):

$$\partial^2 v_i/\partial t^2 - c_0^2\,\partial^2 v_j/\partial x_i\,\partial x_j - (\mu/4\pi\rho)\,(H_j\,\partial/\partial x_j)^2\,v_i$$
$$+ (\mu/4\pi\rho)\,(H_i H_j\,\partial^2 v_k/\partial x_j\,\partial x_k + H_j H_k\,\partial^2 v_j/\partial x_i\,\partial x_k - H^2\,\partial^2 v_k/\partial x_i\,\partial x_k) = 0. \tag{8}$$

If the magnetic field is neglected ($\mathbf{H} = 0$), this equation reduces to the first two terms; its curl then shows that the vorticity $\boldsymbol{\omega} = \nabla \wedge \mathbf{v}$ is conserved in the mean state, i.e. $\partial^2\boldsymbol{\omega}/\partial t^2 = 0$. On taking the divergence $\partial/\partial x_i$, it is concluded that the dilatation $\Delta \equiv \nabla\,.\,\mathbf{v}$ propagates acoustically, according to

$$\{\partial^2/\partial t^2 - c_0^2\,\partial^2/\partial x_i^2\}\,\nabla\,.\,\mathbf{v} = 0, \quad c_0^2 \equiv (\partial p/\partial\rho)_s = \gamma p/\rho, \tag{9}$$

i.e. radially (isotropically) with a phase speed c_0 specified by the adiabatic compressibility of the fluid and proportional to the square root of pressure divided by density. The coefficient γ is the adiabatic exponent, which is $\frac{5}{3}$, $\frac{7}{5}$ and $\frac{4}{3}$ respectively for monatomic, diatomic and polyatomic perfect gases.

If the fluid is regarded as incompressible, i.e. $\nabla\,.\,\mathbf{v} = \partial v_k/\partial x_k = 0$, the propagation equation (8) reduces to the first, third and fifth terms. It follows that the velocity component $\mathbf{v}\,.\,\mathbf{l}$ (with $\mathbf{l} = \mathbf{H}/H$) along the magnetic field is conserved in the mean state, i.e. $\partial^2(\mathbf{v}\,.\,\mathbf{l})/\partial t^2 = 0$. The transverse component $\mathbf{v} \wedge \mathbf{l}$ propagates like an Alfvén wave:

$$\{\partial^2/\partial t^2 - c_1^2\,\partial^2/\partial l^2\}\,(\mathbf{v} \wedge \mathbf{l}) = 0, \quad c_1^2 \equiv \mu\mathbf{H}^2/4\pi\rho = 2P/\rho. \tag{10}$$

That is, $\mathbf{v} \wedge \mathbf{l}$ propagates along magnetic lines of force $\partial/\partial l = l_i\,\partial/\partial x_i$ with velocity proportional to the magnetic field (Alfvén 1942, 1948). The Alfvén speed c_1 corresponds to the speed of sound in that it is proportional to the square root of the *magnetic* pressure $P = \mu H^2/8\pi$ divided by the density. The adiabatic exponent is replaced here by the factor 2, so that the 'magnetic gas' may be identified as a perfect gas whose molecules have $N = 2$ degrees of freedom transverse to the magnetic field, since $\gamma = 1 + 2/N$.

In the general case of a compressible fluid ($\partial v_k/\partial x_k \neq 0$) subjected to a magnetic field ($H_i \neq 0$), the full propagation equation is of the form $\square_{ij}\{v_j\} = 0$. The linear, homogeneous, second-order partial differential operator

$$\square_{ij} \equiv \delta_{ij}\,\partial^2/\partial t^2 - c_0^2\,\partial^2/\partial x_i\,\partial x_j - c_1^2(\delta_{ij}\,\partial^2/\partial l^2 - l_j\,\partial^2/\partial x_i\,\partial l) + c_1^2(l_i\,\partial^2/\partial x_j\,\partial l - \partial^2/\partial x_i\,\partial x_j) \quad (11)$$

describing propagation may be called the *magneto-acoustic wave operator*. It consists of: (i) second-order wave time dependence, allowing both propagating and standing waves; (ii) a term involving the speed of sound c_0 and the dilatation $\Delta = \partial v_j/\partial x_j$, as for acoustic waves; (iii) two terms involving the Alfvén speed c_1 and directed derivative $\partial/\partial l$, as for Alfvén waves; (iv) two terms combining magnetic derivatives $\partial/\partial l$ and the Alfvén speed c_1 with the dilatation $\partial v_j/\partial x_j$, representing magneto-acoustic interaction.

The properties of waves between the contrasting limits of longitudinal, isotropic acoustic waves and transverse, one-dimensional Alfvén waves may be studied by Fourier (series or integral) decomposition of the wave field. The functional form of a component $V_j = a_j \exp\,[i(\mathbf{k}.\mathbf{x} - \omega t)]$, in which $\mathbf{k}$ is the wave vector and ω the frequency, implies that $\square_{ij}(\partial/\partial\mathbf{x},\ \partial/\partial l,\ \partial/\partial t) = -\Pi_{ij}(\mathbf{k},\ \mathbf{kl},\ -\omega)$ specifies, from (11), the magneto-acoustic dispersion matrix

$$\Pi_{ij} = \{(c_0^2 + c_1^2)\,k_i k_j - \omega^2\delta_{ij}\} - c_1^2(\mathbf{k}.\mathbf{l})\,(k_i l_j + k_j l_i - \mathbf{k}.\mathbf{l}\,\delta_{ij}). \quad (12)$$

The propagation equation $\square_{ij}\{v_j\} = 0$ transforms to $\Pi_{ij}\,a_j = 0$, so that the condition $|\Pi_{ij}| = 0$ for the existence of waves of non-zero amplitude ($a_j \neq 0$) defines the dispersion relation $\omega = \omega(\mathbf{k})$.

Choosing the x_1 axis in the direction of the wave normal $\mathbf{n} = \mathbf{k}/k$ and the x_2 axis in the plane containing this vector and the magnetic field $\mathbf{l} = \mathbf{H}/H$ (e.g. Landau & Lifshitz 1959, vol. 8, §52) gives

$$\Pi_{ij} = k^2\begin{pmatrix} c_0^2 + c_1^2(1 - l_1^2) - c_*^2 & -c_1^2 l_1 l_2 & 0 \\ -c_1^2 l_1 l_2 & c_1^2 l_1^2 - c_*^2 & 0 \\ 0 & 0 & c_1^2 l_1^2 - c_*^2 \end{pmatrix}, \quad (13)$$

in which we have introduced the phase velocity $c_* = \omega/k$; since this does not depend on the frequency but on direction alone, magneto-acoustic modes are non-dispersive but anisotropic. The condition $|\Pi_{ij}| = 0$ can be satisfied by $\Pi_{33} = 0$, which defines an Alfvén mode $c_* = c_1\mathbf{l}.\mathbf{x}$ that extends (from incompressible) to compressible fluid and has a velocity perturbation $\mathbf{V}_3$ transverse to both the magnetic field and the wavenumber: $\mathbf{V}_3 = V_3\,\mathbf{n} \wedge \mathbf{l}$.

This is uncoupled from the other two modes, whose perturbations are in the plane containing the direction of propagation and the magnetic field: $\mathbf{V} \in (\mathbf{n}, \mathbf{l})$. Their phase velocities, calculated from $\Pi_{11}\Pi_{22} = \Pi_{12}^2$, are

$$c_\pm = \tfrac{1}{2}\{|c_0\mathbf{n} + c_1\mathbf{l}| \pm |c_0\mathbf{n} - c_1\mathbf{l}|\}, \quad c_+ > c_0 > c_-, \quad (14)$$

and they result from linear superpositions of radial, acoustic waves $c_0\mathbf{n}$ and magnetic, Alfvén waves $c_1\mathbf{l}$ propagating in the same (+) or opposite (−) directions. These modes are coupled, but may be distinguished as *fast* (c_+) or *slow* (c_-) according to whether their phase speed is larger or smaller than the (mathematical not physical) speed of sound c_0.

2.3. *Dynamic and magnetic quadrupoles*

In the consideration of dynamic and magnetic stresses, later to be identified as source or sink quadrupoles, we shall frequently encounter the *symmetric tensor product* of two vectors A_i and B_j, defined by

$$A_i * B_j \equiv A_i B_j + A_j B_i - A_k B_k \delta_{ij}. \tag{15}$$

For example, if the pressure p is separated into a stagnation (constant) part p_0, a dynamic part $\frac{1}{2}\rho v^2$ and a compressible part p', the hydrodynamic stresses $\rho v_i v_j - p\delta_{ij}$ in an inviscid fluid take the form $\frac{1}{2}\rho v_i * v_j + p'\delta_{ij}$. The magnetic stresses are

$$(-\mu/8\pi)(H_i + h_i) * (H_j * h_j),$$

and imply a linear term $(-\mu/4\pi) h_i * H_j$, whose divergence appears in (7c).

This notation is convenient in repeating the process of elimination between the left-hand sides of (7a–c) to form the propagation equation, with the difference that now all terms will be retained, to include dissipation and generation effects in an exact wave equation. The elimination is performed by applying $\partial/\partial t$ to (7c), and substituting for $\partial\rho/\partial t$ and $\partial \mathbf{h}/\partial t$ on the left-hand side from (7a, b). This gives

$$\frac{\partial^2 v_i}{\partial t^2} + \frac{c_0^2}{\rho}\frac{\partial}{\partial x_i}\left(-\rho\frac{\partial v_j}{\partial x_j} - \frac{\partial}{\partial x_j}\rho' v_j\right)$$
$$-\frac{\mu}{4\pi\rho}\frac{\partial H_j}{\partial x_j} * \left\{-H_i\frac{\partial v_k}{\partial x_k} + H_k\frac{\partial v_i}{\partial x_k} + \nabla\wedge(\mathbf{v}\wedge\mathbf{h})_i + \frac{c^2}{4\pi\mu\sigma}\nabla^2\mathbf{h}_i\right\}$$
$$= -\frac{\partial^2}{\partial t^2}\frac{\rho'}{\rho}v_i - \frac{1}{\rho}\frac{\partial^2}{\partial t\,\partial x_j}\left\{\frac{\rho}{2}v_i * v_j + (p' - c_0^2\rho)\delta_{ij} - \tau_{ij} - \frac{\mu}{8\pi}h_i * h_j\right\}. \tag{16}$$

The nonlinear terms of hydrodynamic origin form the *dynamic tensor*

$$R_{ij} \equiv \tfrac{1}{2}\rho v_i * v_j + (p' - c_0^2\rho' - \alpha_0^2 s')\delta_{ij}, \tag{17}$$

which consists of convection stresses $\frac{1}{2}\rho v_i * v_j$ and stresses arising from the nonlinear terms $p' - c_0^2\rho' - \alpha_0^2 s'$ of the equation of state $p = p(\rho, s)$; here $c_0^2 \equiv (\partial p/\partial \rho)_s$ and $\alpha_0^2 \equiv (\partial p/\partial s)_\rho$. The nonlinear terms of magnetic origin form the *magnetic tensor* S_{ij}, given by

$$-\frac{\partial}{\partial t}S_{ij} \equiv \frac{\partial}{\partial t}\left(\frac{\mu}{8\pi}h_i * h_j\right) + \frac{\mu}{4\pi}\nabla\wedge(\mathbf{v}\wedge\mathbf{h})_i * H_j, \tag{18}$$

which consists of the magnetic stresses $-(\mu/8\pi)h_i * h_j$ and stresses due to transverse transport $\mathbf{v}\wedge\mathbf{h}$ of the magnetic field.

If an external force field, e.g. gravity $\mathbf{G}$, were applied, and thus appeared in the momentum equation (5), it would give rise to a term $\rho^{-1}\,\partial G_i/\partial t$ in (16). Similarly a source, e.g. of mass Q, appearing in the equation of continuity as $\partial Q_i/\partial t$, would give a term $-\rho^{-1}\,\partial^2 Q/\partial t\,\partial x_i$. Both would have to be incorporated in the dynamic tensor R_{ij}. The dissipative terms (linear or nonlinear, dynamic or magnetic) can be collected in a similar expression: the *dissipation tensor*, given by

$$\frac{\partial}{\partial t}D_{ij} \equiv -\alpha_0^2\frac{\partial s}{\partial t}\delta_{ij} + \frac{\partial}{\partial t}\tau_{ij} + \frac{c^2}{16\pi^2\sigma}\nabla^2 h_i * H_j, \tag{19}$$

which consists of stresses due to entropy production, viscosity and electrical resistance.

In addition to these there is a nonlinear term in the form of an acoustic operator (the first two terms in (11), to be precise) applied to the nonlinear mass flux $(\rho'/\rho)\,v_j$. This term vanishes for Alfvén waves ($\rho' = 0$), so that it is sufficient to estimate its order of magnitude in the opposite extreme of acoustics. The density fluctuation is $\rho' \sim \rho M_0^2$, where $M_0 \equiv v/c_0$ is the Mach number, and the whole term $\sim \partial^2(M_0^2 v)/\partial t^2$. The dynamic tensor appears as $\rho^{-1}\,\partial^2 R_{ij}/\partial t\,\partial x_j$, with leading term $\sim \rho v^2$ [see (17)], i.e.

$$\frac{1}{\rho}\frac{\partial^2}{\partial t^2}\left(\frac{\rho v^2}{c_0}\right) \sim \frac{\partial^2}{\partial t^2}(M_0 v).$$

Thus the scattering term

$$\left\{\delta_{ij}\frac{\partial^2}{\partial t^2} - c_0^2\frac{\partial^2}{\partial x_i\,\partial x_j}\right\}\rho' v_j \sim O(M_0)\frac{\partial^2 R_{ij}}{\partial t\,\partial x_j} \tag{20}$$

is negligible at Mach numbers $M_0 \equiv v/c_0 \ll 1$ for arbitrary Alfvén number $M_1 \equiv v/c_1$.

The remaining terms are linear and non-dissipative, and constitute the propagation operator $\square_{ij}$ [see (11)] in the *complete magneto-acoustic wave equation*

$$\square_{ij}\{v_j\} - \frac{1}{\rho}\frac{\partial^2 D_{ij}}{\partial t\,\partial x_j} = -\frac{1}{\rho}\frac{\partial^2}{\partial t\,\partial x_j}(R_{ij} + S_{ij}). \tag{21}$$

This consists of (i) the magneto-acoustic wave operator (11) applied to the velocity perturbation, which represents propagation, (ii) the dissipation tensor (19), which acts as a sink quadrupole, and (iii) the dynamic tensor (17) and the magnetic tensor (18), which act as source quadrupoles, modelling generation.

3. The physical process of generation

The interpretation of the dynamic and magnetic tensors as source quadrupoles indicates the order of magnitude of wave generation by turbulence and inhomogeneities. Dissipation by viscosity, heat conduction and electrical resistance acts as a sink of waves, except for the effect of heat production, which acts as a source. The overall magneto-acoustic quadrupole may also be classified by components, according to the modes of emission.

3.1. *Generation by turbulence and inhomogeneities*

The dynamic tensor R_{ij} and the magnetic tensor S_{ij} exhibit formal similarity, at least in the first terms, which suggests that they be considered in parallel. The first term of the dynamic tensor is the convection stress

$$R_{ij}^{(1)} \equiv \tfrac{1}{2}\rho v_i * v_j = \rho v_i v_j - \tfrac{1}{2}\rho v^2\,\delta_{ij}, \tag{22}$$

which consists of anisotropic stresses $\rho v_i v_j$ and the dynamic pressure $\frac{1}{2}\rho v^2$. The first term of the magnetic tensor is the magnetic stress

$$S_{ij}^{(1)} \equiv -\frac{\mu}{4\pi}h_i * h_j = -\frac{\mu}{4\pi}h_i h_j + \frac{\mu}{8\pi}h^2\,\delta_{ij}, \tag{23}$$

which consists of anisotropic stresses $-(\mu/4\pi)\,h_i h_j$ and the magnetic pressure $\mu h^2/8\pi$.

Both expressions consist of anisotropic stresses and a pressure term quadratic in their respective variables, the velocity perturbation **v** and the magnetic-field perturbation **h**. These are larger in the regions of hydromagnetic turbulence, which from the point of view of generation of magneto-acoustic waves are modelled by the quadrupoles (22) and (23). An example is the generation of waves in the lower photosphere of the sun, whose propagation into the chromosphere and subsequent dissipation could explain (e.g. Lighthill 1967) the rise in temperature with altitude.

The second term

$$R_{ij}^{(2)} \equiv (p' - c_0^2 \rho' - \alpha_0^2 s')\,\delta_{ij} \tag{24}$$

of the dynamic tensor is the deviation from isentropic, homogeneous acoustics; it is non-zero if the speed of sound c_0 (or α_0) is not uniform, e.g. because of variations in density. The second term of the magnetic tensor, given by

$$\frac{\partial}{\partial t} S_{ij}^{(2)} \equiv -\frac{\mu}{4\pi}\{\nabla \wedge (\mathbf{v} \wedge \mathbf{h})_i H_j + H_i \nabla \wedge (\mathbf{v} \wedge \mathbf{h})_j - \mathbf{H}.\nabla \wedge (\mathbf{v} \wedge \mathbf{h})\,\delta_{ij}\}, \tag{25}$$

represents the deviation from homogeneous Alfvén waves in a perfectly conducting medium, for which the velocity and magnetic-field perturbations are parallel, i.e. $\mathbf{v} \wedge \mathbf{h} = 0$. It is non-zero if the Alfvén speed c_1 is not constant, e.g. because the external magnetic field **H** is non-uniform.

Both second terms represent deviations from physical or constitutive homogeneity of the fluid flow, e.g. variations in density or in the external magnetic field. These are more important in regions containing fluid or flow inhomogeneities, which from the point of view of generation of magneto-acoustic waves are modelled by (24) and (25). Examples are solar spots (e.g. Alfvén 1943) and also the rims of interstellar gas clouds, which stand out in observations from the earth because of their brightness.

The order of magnitude of inhomogeneous sources can be comparable to that of turbulent sources, whose dynamic part (22) and magnetic part (23) scale as

$$R \sim \tfrac{1}{2}\rho U^2, \quad S \sim (\mu/8\pi) H^2 \tag{26}$$

respectively on the dynamic and magnetic pressures. These are evaluated in the source regions, where the large perturbations in the velocity U and magnetic field H may be comparable to mean-flow values. The equation evinces the physical congruences of generation

$$U \leftrightarrow H, \quad \rho \leftrightarrow \mu/4\pi, \tag{27a, b}$$

the velocity corresponding to the magnetic field and the mass density ρ to the magnetic permeability μ (divided by 4π), which is likened to a 'magnetic mass'.

In a flow in which (27*a*, *b*) were equalities and the velocity **v** and magnetic field **h** were parallel, the dynamic and magnetic sources (23) and (24), which have opposite signs, would cancel. In hydromagnetic turbulence (Batchelor 1950) the magnetic field **h** tends to be parallel to the vorticity $\nabla \wedge \mathbf{v}$, so that a weakening of turbulent sources would occur in Beltrami flows, in which **v** is parallel to $\nabla \wedge \mathbf{v}$. An example is the hydromagnetic flow in the mantle of the earth, which should be stable (Moffatt 1976), owing to the relative constancy of the earth's magnetic field.

The relative importance of magnetic and dynamic sources of magneto-acoustic waves is indicated by the ratio of the respective pressures

$$\eta \equiv S/R \sim (\mu/4\pi\rho)(H/U)^2, \tag{28}$$

which equals the product of the ratio of the 'masses' in (27 *b*) and the squared variables. If $\eta \ll 1$ the dynamic source R_{ij} predominates, if, instead, $\eta \gg 1$ the magnetic source S_{ij} is more important, and for $\eta \sim 1$ they are comparable. The condition of non-negligible magnetic pressure $\eta \gtrsim 1$, used to decide when a hydrodynamical study should be extended to magnetohydrodynamics, implies the possible existence of non-negligible magnetic wave sources.

3.2. *Dissipation through diffusion and resistance*

Dissipation in a magnetohydrodynamic flow is specified by the equation of energy (Landau & Lifshitz 1959, vol. 8, §51)

$$\rho T \frac{ds}{dt} = \tau_{ij}\, \partial v_i/\partial x_j + \kappa \nabla^2 T + \frac{c^2}{16\pi\mu\sigma}(\nabla \wedge \mathbf{H})^2, \tag{29}$$

which states that release of heat can occur owing to (i) viscous stresses τ_{ij} in a shear flow $\partial v_i/\partial x_j$, (ii) conduction of heat κ in a non-uniformly heated ($\nabla T \neq 0$) fluid and (iii) electrical resistance $1/\sigma$ to currents $\mathbf{J}$ (Joule effect J^2/σ) in the fluid due to the non-uniform magnetic field [see (1)]. Since the mean state is one of rest ($ds/dt = \partial s/\partial t$), the first term in the expression (19) for the dissipation tensor becomes

$$\frac{\partial}{\partial t} D_{ij} = \left(\frac{\partial}{\partial t}\tau_{ij} - \tau_{kl}\frac{\partial v_k}{\partial x_l}\beta\delta_{ij}\right) - \kappa\nabla^2 T\beta\delta_{ij} + \frac{c^2}{16\pi^2\sigma}\{\nabla^2 h_i * H_j - (\nabla\wedge\mathbf{h})^2\beta\delta_{ij}\}, \tag{30}$$

with $\beta \equiv \alpha_0^2/\rho T$, showing separately the effects of viscosity, heat conduction and electrical resistance.

The linear approximation to the viscous stresses (Landau & Lifshitz 1959, vol. 6, §15),

$$\tau_{ij} = \nu(\partial v_i/\partial x_j + \partial v_j/\partial x_i) + (\nu' - \tfrac{2}{3}\nu)\, \partial v_k/\partial x_k\, \delta_{ij}, \tag{31}$$

shows that they are due to the resistance of the fluid to shearing ($\partial v_i/\partial x_j + \partial v_j/\partial x_i$) and compressive ($\partial v_k/\partial x_k$) motion, with viscosities ν and ν'. The corresponding magnetic quantity is the *current stress tensor*, defined by

$$\zeta_{ij} \equiv (c^2/16\pi^2\sigma)\,(H_i\nabla^2 h_j + H_j\nabla^2 h_i - \mathbf{H}.\nabla^2\mathbf{h}\delta_{ij}). \tag{32}$$

Noting from (1) and $\nabla.\mathbf{h} = 0$ that $\nabla^2\mathbf{h} = -\nabla\wedge(\nabla\wedge\mathbf{h}) = -(4\pi/c)\nabla\wedge\mathbf{J}$ and $\mathbf{H}.\nabla^2\mathbf{h} = (4\pi/c)\nabla.(\mathbf{H}\wedge\mathbf{J})$, where $\mathbf{J}$ is the electric current, (32) specifies the stresses in the resistive medium ($\zeta \equiv 1/\sigma \neq 0$) due to longitudinal ($\mathbf{J}.\mathbf{H}$) and transverse ($\mathbf{J}\wedge\mathbf{H}$) currents.

The expression (30) for the dissipation tensor involves $\beta \equiv \alpha_0^2/\rho T$, in which $\alpha_0^2 \equiv (\partial p/\partial s)_\rho$, so that $\beta = (\rho T)^{-1}(\partial p/\partial T)_\rho/(\partial s/\partial T)_\rho$. From the definition of the specific heat at constant volume, $C_v \equiv T(\partial s/\partial T)_\rho$, the thermodynamic factor may be calculated as $\beta = (\partial p/\partial T)_\rho/\rho C_v$ for an equation of state of the form $p = p(\rho, T)$. For example, for a perfect gas $p = \rho RT$, so $\beta = R/C_v$ is a constant (Landau & Lifshitz 1959, vol. 5, chap. IV), respectively equal to $\frac{2}{3}$, $\frac{2}{5}$ and $\frac{1}{3}$ for monatomic, diatomic and polyatomic molecules. As it has been derived from the entropy production terms, we may designate β the *heat release factor*, and, since it appears with a minus sign in dissipation terms, it actually acts as a source. Thus a source of heat of strength W would give rise to an additional source $-(\beta/\rho)\,\partial^2 W/\partial t\,\partial x_i$ in the (right-hand side of the) complete wave equation (21).

The magnitude of viscous stresses may be estimated from (31) as $\sim \nu U/L$; however dissipation D_ν by viscosity occurs both directly as a sink in (30) and indirectly, through heat release, as a source, resulting in the factor $1-\beta$. Dissipation D_κ by heat conduction is of this order divided by the Prandtl number $Pr \equiv \nu C_p/\kappa$ (Landau & Lifshitz 1959, vol. 6, §53), where C_p is the specific heat at constant pressure and κ the thermal conductivity; this is a process of heat release, acting as source, with coefficient $-\beta$. Electrical resistance resembles viscous resistance in duality: direct dissipation and indirect generation occur, resulting in the factor $1-\beta$ in D_σ. In the estimation of the current stresses $c^2H^2/16\pi^2\sigma L^2$ in (32) a factor L/U appears because this is a term in $\partial D_{ij}/\partial t$. Thus

$$\frac{D_\nu}{1-\beta} \sim \nu\frac{U}{L}, \quad \frac{D_\kappa}{-\beta} \sim \frac{\kappa}{C_p}\frac{U}{L}, \quad \frac{D_\sigma}{1-\beta} \sim \frac{c^2}{16\pi^2}\frac{1}{\sigma}\frac{H^2}{UL}. \tag{33}$$

The orders of magnitude of dissipation by viscosity, heat conduction and electrical resistance, normalized respectively by the dynamic and magnetic source strengths (26), are given by

$$\frac{D_\nu/R}{1-\beta} \sim \frac{\nu}{\rho UL} \equiv Re^{-1}, \quad \frac{D_\kappa/R}{-\beta} \sim \frac{\kappa/C_p}{\rho UL} \equiv Pe^{-1}, \quad \frac{D/S}{1-\beta} \sim \frac{c^2/4\pi\mu}{\sigma UL} \equiv Me^{-1}. \tag{34}$$

These define (apart from a source/sink factor) the inverses of the Reynolds number Re, the Péclet number Pe and the magnetic Reynolds number Me (Lehnert 1952), which indicate (without the common UL term) the dissipation congruences: the dynamic viscosity ν/ρ corresponds to the thermal diffusivity $\kappa/\rho C_p$ and the effective resistivity $c^2/4\pi\mu\sigma$. These act as diffusion parameters χ in a $\chi\,\partial v_i/\partial t$ term in the complete wave equation, causing a decay in the amplitude of propagating waves that is considerable (e^{-1}) over distances $d \sim c_*/\chi$, where c_* is the phase speed.

In the generation region of large perturbations, dissipation could be significant in reducing these over moderate scales, damping the wave sources in the manner of a sink. The processes of dissipation would overwhelm generation for sufficiently low hydrodynamic and magnetic Reynolds numbers: Re, $Me < 1-\beta$. Heat production $\beta/Pe > (1-\beta)(Re^{-1}+Me^{-1})$ could still emerge as a source for small Péclet number, $\beta > (Re^{-1}+Me^{-1})/(Re^{-1}+Me^{-1}+Pe^{-1})$, corresponding to a predominance of heat conduction. Otherwise, if dissipation were still considerable, it could be accounted for by factors $1-(D_\nu+D_\kappa)/R$ and D_σ/S, which respectively reduce the dynamic and magnetic source strengths (26) to the effective values

$$\bar{R} \sim \left(1+\frac{\beta}{Pe}-\frac{1-\beta}{Re}\right)R, \quad \bar{S} \sim \left(1-\frac{1-\beta}{Me}\right)S. \tag{35}$$

3.3. *Classification of emission components*

The dynamic and magnetic tensors have been shown to be quadratic in their respective variables, while the dissipation tensor is a function of their derivatives:

$$R \propto v^2, \quad S \propto h^2, \quad D \propto \partial(v,h). \tag{36}$$

Thus the magnetic generation and dissipation of waves is caused by the perturbation $\mathbf{h}$ from the constant external magnetic field $\mathbf{H}$. Similarly, dynamic generation and dissipation would be due to the perturbation velocity $\mathbf{v}$, even if the mean state were in uniform convection with velocity $\mathbf{V}$.

In the latter case the preceding inferences would remain valid in a frame moving with the mean flow; in returning to observations from rest, the velocity would be transformed to $v' = \mathbf{V}+\mathbf{v}$. Expressions quadratic in v' would be similar in terms of v, and derivatives of $\mathbf{v}'$ equal those of $\mathbf{v}$ (because $\mathbf{V}$ is constant), leaving unaltered generation and dissipation. However, scattering terms bilinear in $\mathbf{v}$ and $\mathbf{V}$ could appear, and in the wave operator (11) the local time derivative would include a convection effect and become the material derivative $d/dt \equiv \partial/\partial t+\mathbf{V}.\nabla$.

The interpretation of nonlinear and dissipative terms in the exact wave equation as sources or sinks, which was the conceptual advance of the original 'acoustic analogy' (Lighthill 1952), is simpler when turbulence and inhomogeneities are contained in a small region D. The stresses T can be calculated from the perturbation flow $\mathbf{v}'$ in D, and taken as 'model' sources generating the wave field v propagating outside D:

$$v = F\{T(\mathbf{v}')\}. \tag{37}$$

If the source region is multiply connected (e.g. separating positively and negatively charged clouds in an electric field) or non-compact (i.e. extensive on the scale of the wavelength), the wave functional F may be complicated by scattering effects.

To study the interaction between the sources and the waves they generate (*in D*), the basic analogy (37) could be taken as the initial step in an iterative approximation

$$\mathbf{v}_{n+1} = F_*\{T(\mathbf{v}'+\mathbf{v}_n)\} = F_*\{T(\mathbf{v}'+F_*\{T(\mathbf{v}'+\ \dots)\})\}. \tag{38}$$

The nth estimate $\mathbf{v}_n$ of the wave field is superimposed on the perturbation $\mathbf{v}'$ before computing the 'model source' of the following estimate $\mathbf{v}_{n+1}$ of the wave field. The near-field wave functional F_*, valid in D, would be used, in which case the operator series (38) should converge, provided the wave generation process is stable.

The overall source/sink of magneto-acoustic waves is the sum of the dynamic and magnetic tensors, which act as sources, minus the dissipation tensor, which thus acts as a sink:

$$T_{ij} \equiv R_{ij}+S_{ij}-D_{ij}. \tag{39}$$

This is the *magneto-acoustic quadrupole*, which appears [see (21)] in the complete wave equation

$$\Box_{ij}\{v_j\} = -\rho^{-1}\,\partial^2 T_{ij}/\partial t\,\partial x_j \tag{40}$$

as the forcing term of the propagation expression $\Box_{ij}\{v_j\} = 0$.

It has been mentioned that there exist three modes of magneto-acoustic propagation (§2.2): a slow and a fast mode with perturbations in the plane of the direction of propagation $\mathbf{n}$ and the magnetic field $H\mathbf{l}$, i.e. $v_a \in (\mathbf{n},\mathbf{l})$ with $a = 1, 2$, and an uncoupled, transverse Alfvén mode v_3. Also the propagation operator (written in these coordinates) consists of a 'square' term $\Box_{ab}$ $(b = 1,2)$ plus a 'corner' term $\Box_{33}$ [$\Box_{3a} = \Box_{a3} = 0$, as in (13)]. Thus $\Box_{ij}\{v_j\}$ separates into $\Box_{ab}\{v_b\}$ for slow and fast modes and $\Box_{33}\{v_3\}$ for the Alfvén mode, so that (40) may be decomposed into

$$\Box_{ab}\{v_b\} = -\rho^{-1}(\partial^2 T_{ab}/\partial t\,\partial x_b+\partial^2 T_{a3}/\partial t\,\partial x_3), \tag{41a}$$

$$\Box_{33}\{v_3\} = -\rho^{-1}(\partial^2 T_{3b}/\partial t\,\partial x_b+\partial^2 T_{33}/\partial t\,\partial x_3), \tag{41b}$$

which shows that the former modes are generated by the components $T_{ai} \equiv (T_{ab}, T_{a3})$ and the latter by $T_{3i} \equiv (T_{3b}, T_{33})$. Since T_{ij} is symmetric, the components $T_{a3} = T_{3a}$ are the only ones that can actually generate all modes.

Thus the components of the magneto-acoustic quadrupole may be classified according to the modes of emission by the scheme

$$T_{ij} = \begin{bmatrix} \begin{matrix} T_{ab} \\ \text{(slow, fast)} \end{matrix} & \begin{matrix} T_{a3} \\ \text{(all)} \end{matrix} \\ \begin{matrix} T_{3b} \\ \text{(all)} \end{matrix} & \begin{matrix} T_{33} \\ \text{(Alfvén)} \end{matrix} \end{bmatrix}, \quad a, b \in (1,2), \tag{42}$$

as follows: (i) the components T_{ab} in the plane of the wave vector $\mathbf{n}$ and the magnetic field $H\mathbf{l}$ emit slow and fast modes $v_a \in (\mathbf{n}, \mathbf{l})$; (ii) the transverse component T_{33} emits the Alfvén mode $\mathbf{v}_3 = v_3 \mathbf{n} \wedge \mathbf{l}$; (iii) the mixed components $T_{a3} = T_{3a}$ emit all three (slow, fast and Alfvén) modes.

4. The limiting properties of radiation

The formal and physical description of general magneto-acoustic waves should be consistent with the theory of aerodynamic acoustics (Lighthill 1952) when the magnetic field is neglected, and should yield a corresponding theory for Alfvén waves if, instead, compressibility is ignored. Some of these particular results should remain valid respectively in the limits of weak magnetic field (hydrodynamics) and near incompressibility (hydromagnetics), though greater contrast should be expected in the opposite limit of strong magnetic field (magnetodynamics). The asymptotic approximation to the wave field evinces the directivity and allows a dimensional estimate of the intensity of magneto-acoustic radiation.

4.1. '*Acoustic*' and '*Alfvén*' analogies

If the magnetic field is neglected ($\mathbf{H} = 0$) the magnetic tensor S_{ij} and the magnetic dissipative term in D_{ij} [see (19)] vanish; the hydrodynamic dissipative terms and the dynamic tensor R_{ij} [see (17)], which form the total quadrupole (39), reduce to

$$T_{ij}^{(0)} = \rho v_i v_j + (p - c_0^2 \rho)\, \delta_{ij} - \tau_{ij}, \tag{43}$$

known as the Lighthill tensor (Lighthill 1952). The condition $\mathbf{H} = 0$ also reduces the propagation expression (11) to its first two terms, whose form is similar to that of the scattering effect (20); both are included in the complete wave equation (2.1):

$$\left\{\delta_{ij} \frac{\partial^2}{\partial t^2} - c_0^2 \frac{\partial^2}{\partial x_i\, \partial x_j}\right\} (\rho v_j + \rho' v_j) = \partial^2 T_{ij}^{(0)} / \partial t\, \partial x_j, \tag{44}$$

in which no approximations have been made.

The exact wave equation (44) suggests that the source of acoustic waves is the Lighthill tensor (43). The wave variable is the velocity perturbation $\mathbf{v}$, but since in the propagation region all variables are proportional, e.g. $\rho'/\rho = v/c$ (Landau & Lifshitz 1959, vol. 6, §63), their model sources must be equivalent. However this statement may become clearer if the exact equation of continuity (7*b*) is used to re-express (44) with the mass density ρ as the wave variable:

$$\left\{\frac{\partial^2}{\partial t^2} - c_0^2 \frac{\partial^2}{\partial x_i^2}\right\} \rho = \frac{\partial^2 T_{ij}^{(0)}}{\partial x_i\, \partial x_j}, \tag{45}$$

which is in the form of the original 'acoustic analogy' (Lighthill 1952), on which aerodynamic acoustics is based.

If the fluid is regarded as incompressible (a condition equivalent to $d\rho/dt = 0$, $\nabla . \mathbf{v} = 0$, $\rho' = 0$ or $c_0 = \infty$) but is subjected to a magnetic field $\mathbf{H} \neq 0$, a corresponding theory could be obtained for Alfvén waves, based on an 'Alfvén analogy'. The term of the propagation operator associated with compressibility can be cancelled in the hydrodynamic tensor (43), which reduces to

$$T^{(0)\prime}_{ij} = \rho v_i v_j + p\delta_{ij} - \tau_{ij}, \tag{46}$$

an *incompressible* Lighthill tensor. The scattering term that remains, namely the first term in (20), vanishes because $\rho' = 0$.

All magnetic source/sink terms remain, those in the dissipation tensor D_{ij} [see (19)] being collected with the magnetic tensor S_{ij} [see (18)] to give

$$-\frac{\partial}{\partial t} T^{(1)}_{ij} = \frac{\partial}{\partial t}\frac{\mu}{8\pi} h_i * h_j + \frac{\mu}{4\pi} \nabla \wedge (\mathbf{v} \wedge \mathbf{h})_i * H_j + \frac{c^2}{16\pi^2\sigma} \nabla^2 h_i * H_j, \tag{47}$$

which could be designated the *Alfvén tensor*. This is because the propagation expression reduces to Alfvén's expression (10), so that the wave equation (21) simplifies to

$$\left\{\frac{\partial^2}{\partial t^2} - c_0^2 \frac{\partial^2}{\partial l^2}\right\} v_i = -\frac{1}{\rho}\frac{\partial^2}{\partial t\, \partial x_j} (T^{(0)\prime}_{ij} + T^{(1)}_{ij}). \tag{48}$$

This complete *Alfvén wave equation* shows that the model sources of Alfvén waves consist of a hydrodynamic and a hydromagnetic part: the incompressible Lighthill tensor $T^{(0)\prime}_{ij}$ and the Alfvén tensor $T^{(1)}_{ij}$ respectively.

A theory of Alfvén waves in fluids is thus analogous to aerodynamic acoustics, with the formal substitutions

$$(c_0, \partial/\partial x_i, T^{(0)}_{ij}) \leftrightarrow (c_1, \partial/\partial l, T^{(1)}_{ij} + T^{(0)\prime}_{ij}), \tag{49}$$

which imply that (i) the speed of sound c_0 [see (9)] corresponds to the Alfvén speed c_1 [see (10)], (ii) radial (or isotropic) derivatives $\partial/\partial x_i$ correspond to derivatives along magnetic lines of force $\partial/\partial l$ and (iii) the Lighthill quadrupole $T^{(0)}_{ij}$ corresponds to the sum of the incompressible Lighthill quadrupole $T^{(0)\prime}_{ij}$ and the (complete) Alfvén quadrupole $T^{(1)}_{ij}$.

4.2. *Hydrodynamics, hydromagnetics and magnetodynamics*

In the acoustic limit we have neglected the magnetic field $\mathbf{H}$ altogether, a condition expressed, in view of the congruences (27*a, b*), by $\{\mu/4\pi\rho\}^{\frac{1}{2}} H \ll U$. If this restriction, which from (28) is equivalent to $\eta^{\frac{1}{2}} \ll 1$, is relaxed somewhat to $\eta \ll 1$, a weak, but non-negligible, magnetic field $H \sim \eta^{\frac{1}{2}} U$ could exist. The magnetic pressure is still negligible compared with the dynamic pressure, so that the results of hydrodynamics still hold and, in particular, the Lighthill tensor $T^{(0)}_{ij}$ remains the sole source of waves, a hydrodynamic quadrupole. These are the conditions in the lower atmosphere of the earth, in rivers, basins and oceans, and in most types of machinery either driven by or driving fluids (e.g. hydraulic turbines and jet engines, respectively).

Noting that $(c_1/c_0)^2 = (\mu/4\pi\rho)(H/c_0)^2 = \eta M_0^2$, where $M_0 = U/c_0$ is the Mach number, the phase speeds (14) may be expanded in power series. Thus at $O(\eta M_0^2)$,

$$c_+ = c_0, \quad c_- = c_1 \mathbf{l} . \mathbf{n}, \tag{50}$$

which shows that (provided the Mach number is not high) the fast mode is identified with acoustic waves, while the slow and Alfvén modes coincide. The dominant fast mode inherits all the properties of acoustic waves, as in (45). There also exist *weak* Alfvén waves given by

$$\left\{\frac{\partial^2}{\partial t^2}-c_0^2\frac{\partial^2}{\partial l^2}\right\}v_i=-\frac{\eta^{\frac{1}{2}}}{\rho}\,\partial^2 T_{ij}^{(0)}/\partial t\,\partial x_j. \tag{51}$$

These are generated by the hydrodynamic quadrupole $T_{ij}^{(0)}$ but are of small amplitude ($\propto \eta^{\frac{1}{2}}$).

In the Alfvén-wave limit the fluid was considered virtually incompressible, a condition expressed by $c_1/c_0 \ll 1$. If this restriction is relaxed to $(c_1/c_0)^2 \ll 1$ (hydromagnetics) the identification of modes (50) is preserved. Because the Mach number M_0 is small (c_0 is large), the equivalent condition $\eta M_0^2 \ll 1$ permits the magnetic sources $T_{ij}^{(1)}$ to be significant, as well as the incompressible hydrodynamic quadrupole $[(c_0/c_1)^2=(\eta M_0^2)^{-1}\gg 1]$. Examples include nearly incompressible but ionizable liquids, such as sodium and potassium solutions, used in heat exchanger circuits of nuclear reactors, and also mercury in some laboratory experiments. The coincident slow and Alfvén modes retain the properties of the latter, synthesized in (48). Fast modes, similar to *weak* acoustic waves, also exist, and are given by

$$\left\{\frac{\partial^2}{\partial t^2}-c_0^2\frac{\partial^2}{\partial x_i^2}\right\}v_i=-\frac{1}{\rho M_0}\frac{\partial^2}{\partial t\,\partial x_j}(T_{ij}^{(1)}+T_{ij}^{(0)\prime}), \tag{52}$$

being generated by the same sources as Alfvén waves, but having a small $[O(M_0)]$ amplitude.

Greater contrast both with acoustic and with Alfvén waves could result if the magnetic pressure dominated over the dynamic pressure ($\eta \gg 1$). This is a situation opposite to both hydrodynamics and hydromagnetics in that the motion of the fluid is determined only by the strong magnetic field to which it is subjected (*magnetodynamics*). Current research on controlled nuclear fusion is based on attempts to 'pinch' a plasma in a magnetic field, though for most geometries experimented with so far (e.g. toroidal) wavelike instabilities develop in a way that allows the plasma to break confinement in a fraction of a second, the energy thus produced being still short of the hoped for 'fusion reactor'.

In the stated approximation ($\eta \gg 1$), the sources of waves are magnetic, forming the *magnetodynamic quadrupole* $T_{ij}^{(1)}$ [see (47)]. The condition $(c_0/c_1)^2=(\eta M_0^2)^{-1}\ll 1$ allows expansion of the phase speeds (14) in power series, and to $O(\eta^{-1}M_0^{-2})$,

$$c_+=c_1,\quad c_-=c_0\mathbf{l}.\mathbf{n}, \tag{53}$$

showing that, because the Alfvén speed is larger than the speed of sound, the former corresponds to the fast mode and the latter to the slow mode. The direction of propagation of the modes has not changed, so that the Alfvén waves propagate radially and the acoustic waves along magnetic lines of force. Only the Alfvén mode remains unaffected in speed and direction, but its sources are purely magnetic, i.e. $T_{ij}^{(1)}$.

Reviewing the three magneto-acoustic wave modes in the magnetodynamic limit, we conclude that (i) the slow mode propagates at the speed of sound along magnetic lines of force, i.e.

$$\{\partial^2/\partial t^2-c_0^2\,\partial^2/\partial l^2\}\,v_i=-\rho^{-1}\,\partial^2 T_{ij}^{(1)}/\partial t\,\partial x_j, \tag{54}$$

(ii) the fast mode propagates spherically at the Alfvén speed, i.e.

$$\{\partial^2/\partial t^2-c_1^2\,\partial^2/\partial x_i^2\}\,v_i=-\rho^{-1}\partial^2 T_{ij}^{(1)}/\partial t\,\partial x_j, \tag{55}$$

and (iii) the Alfvén mode propagates at the Alfvén speed along magnetic lines of force, i.e.

$$\{\partial^2/\partial t^2 - c_1^2\,\partial^2/\partial l^2\}\,v_i = -\rho^{-1}\,\partial^2 T_{ii}/\partial t\,\partial x_j. \tag{56}$$

The sources of all modes are the turbulent, inhomogeneous and current stresses, all of magnetic origin, forming the magnetodynamic quadrupole $T_{ij}^{(1)}$ [see (47)].

4.3. *Intensity and directivity of radiation*

The magneto-acoustic wave equation (40) (with ρ^{-1} included in T_{ij}) is transformed by Fourier analysis into the linear inhomogeneous system $\Pi_{ij}\,\hat{v}_j = \omega k_j \hat{T}_{ji}$, in which Π_{ij} is the dispersion matrix (12) and a caret denotes the Fourier transform; multiplication by the inverse dispersion matrix $\Pi_{ij}^{-1} \equiv \Lambda_{ij}/\Pi$, where Λ_{ij} is the co-factor of Π_{ij} and $\Pi = |\Pi_{ij}|$, gives the solution for the Fourier component $\hat{v}_i$. If we consider each frequency component of the source separately, the wave field becomes a Fourier integral over a wave vector space d^3k:

$$v_i(\mathbf{x}, t) = \int_{-\infty}^{+\infty} \frac{1}{\Pi}\,\omega k_j \hat{T}_{jn} \Lambda_{ni} \exp(i\mathbf{k}\,.\,\mathbf{x})\,d^3k. \tag{57}$$

In order to calculate the field radiated in a direction $\mathbf{n} = \mathbf{k}/k$ it is simpler (Lighthill 1960) to align the x_1 axis with $\mathbf{n}$, so that the phase factor reduces to $\exp(ik_1x_1)$. The amplitude function $\omega k_j \hat{T}_{jn} \Lambda_{ni}$ for a source of finite duration and extent is regular in the whole complex k_1 plane. The only poles come from $\Pi = |\Pi_{ij}| = 0$, i.e. the condition (§2.2) specifying the magneto-acoustic dispersion relation $\omega = \omega(\mathbf{k})$, which means that the total radiation field is the sum Σ_s of the contributions from each mode of propagation.

Since the poles lie on the real axis $\operatorname{Im} k_1 = 0$, the path of integration L is distorted from this axis to pass above the poles for which $\partial\omega/\partial k_1 < 0$, representing waves coming from infinity, and below the poles satisfying $\partial\omega/\partial k_1 > 0$, corresponding to waves emitted by the source. So, when the path L is closed by a semicircle of large radius in the upper half-plane, the domain D enclosed contains only the latter poles, thus meeting the radiation condition. The integration over k_1 in (57) thus yields $2\pi i$ times the residues at these poles k_{1s}:

$$v_i = 2\pi i\,\Sigma_s \int_\Omega \frac{\omega k_j \hat{T}_{jn} \Lambda_{ni}}{\partial\Pi/\partial k_1} \exp(ik_1x_1)\,dk_2\,dk_3, \tag{58}$$

leaving a $dk_2\,dk_3$ integration over the wavenumber surface Ω defined by

$$k_{1s} = k_1(k_2, k_3;\,\omega).$$

If the wavenumber surface is plane, as for an Alfvén mode, further evaluation of (58) requires knowledge of the Fourier spectrum $\hat{T}_{ij}$ of the source. If the wavenumber surface were singly or doubly curved (as for cylindrical and spherical waves, respectively), the latter being the case of slow and fast modes (e.g. acoustic waves), one or both integrals may be evaluated by the method of stationary phase. The main contribution to the far field, to $O(x^{-\frac{1}{2}})$, comes from the points $d(\mathbf{k}\,.\,\mathbf{x}) = 0$ on the wavenumber surface where the normal $\partial\omega/\partial\mathbf{k}$ lies in the direction $\mathbf{r} = \mathbf{x}/|\mathbf{x}|$ of observation. In the neighbourhood of a stationary point $\mathbf{k}_i'$, the wavenumber surface can be approximated by a quadric

$$k_{1s} = k_1' + \tfrac{1}{2}\sum_{a=2}^{3} g_a(k_a - k_a')^2 \tag{59}$$

centred at k_i', with principal curvatures g_a.

This introduces in (58) two Gaussian integrals, whose evaluation shows that the wave field

$$v_i = \frac{i4\pi^2}{x_1}\Sigma'_s \frac{k_j \hat{T}_{nj}\Lambda_{ni}}{|g_2 g_3|^{\frac{1}{2}}\, \partial\omega/\partial k_1} \exp\left\{ik_1 x_1 + \tfrac{1}{4}\pi \sum_{a=2}^{3} \operatorname{sgn} g_a\right\} \tag{60}$$

propagates over the $4\pi^2$ angular portion of the unit sphere defined by the curvatures $|g_1 g_2|^{-\frac{1}{2}}$ as a conical pencil (Lighthill 1960) decaying inversely with distance, i.e. as $1/x_1$. A change of sign of the double or single curvature, e.g. when a convergent beam becomes divergent after passing through a focus, causes a 'phase jump' of $\Delta\phi = \frac{1}{2}\pi$ (Landau & Lifshitz 1959, vol. 2, §§ 53, 59) or $\Delta\phi = \frac{1}{4}\pi$ respectively.

To determine the radiation field in a Cartesian system, whose x_1 axis need not be aligned with the direction $\mathbf{r}$ of observation, (60) can be rendered rotationally invariant by the substitutions $x_1 \to r \equiv |\mathbf{x}|$, $k_1 x_1 \to \mathbf{k}.\mathbf{x}$, $\partial\Pi/\partial k_1 \to |\partial\Pi/\partial\mathbf{k}|$ and $g_1 g_2 \to g$, the Gaussian curvature. The asymptotic magneto-acoustic wave field

$$v_i(\mathbf{x}, t) = \Sigma'_s \frac{i4\pi^2}{r} \frac{\omega k_j \hat{T}_{jn}\Lambda_{ni}}{|g|^{\frac{1}{2}}.|\partial\Pi/\partial\mathbf{k}|_s} \exp\{i(\mathbf{k}.\mathbf{x}+\phi)\} \tag{61}$$

has the following properties: (i) the field is radiated by all the stationary points $\mathbf{k}'$ of each mode $|\partial\Pi/\partial\mathbf{k}|_s$, Σ'_s; (ii) it decays three-dimensionally as $1/r$ and is radiated conically through a $4\pi^2$ radian sector with aperture specified by the Gaussian curvature $|g|^{-\frac{1}{2}}$; (iii) it involves the Fourier spectrum of the source $\hat{T}_{jn}$, the dispersion properties of the mode Λ_{ni}, and the emission phase $\mathbf{k}.\mathbf{x}$; (iv) the focal phase ϕ is $\phi = 0$ for anticlastic ($g = g_1 g_2 < 0$) beams, while for synclastic ($g > 0$) beams it is $\phi = \frac{1}{2}\pi$ for divergent ($g_a > 0$) and $\phi = -\frac{1}{2}\pi$ for convergent ($g_a < 0$) beams.

The order of magnitude of the wave far field can be estimated by noting that $\Lambda_{ni}/\Pi \sim \Pi_{ni}^{-1} \sim \omega^{-2}$ [see (12)], that $|g|^{\frac{1}{2}} = |g_2 g_3|^{\frac{1}{2}} \sim k^{-1}$ and that $\hat{T} \sim \bar{T}L^3$ is the total source strength, which is spread with mean density $\bar{T}$ over the domain or scale L^3. The frequency of emission is characteristic of a flow velocity U, thus $\omega \sim U/L$ and $k \sim \omega/c_* \sim U/Lc_*$, and the source strength $\bar{T} \sim (R+S)/\rho$ [see (39), (40)] has both dynamic and magnetic contributions. Thus the dynamic part of the wave far field

$$v \sim (L/r)\,(U^2/c_*^3)\,\bar{T}, \quad \bar{T} \sim U^2 + (\mu/4\pi\rho)\,H^2, \tag{62}$$

scales as in aerodynamic acoustics ($\sim U^4$, Lighthill 1952), but the magnetic term differs in possible directional effects c_*^{-3}, in the ratio of the 'magnetic' and mechanical masses ($\mu/4\pi\rho$), and in the symmetrical dependence on velocity and magnetic perturbations ($\sim U^2H^2$).

The energy flux $\mathbf{E} \sim \mathbf{v}P_T$ is proportional to the total pressure P_T, which consists of the dynamic and magnetic pressures; the equation of momentum $\nabla P_T \sim \rho\, d\mathbf{v}/dt$ implies that $P_T \sim \rho\mathbf{v}.d\mathbf{x}/dt$, so that the energy flux $\mathbf{E} \sim \rho v^2\mathbf{u}$ has the magnitude of the wave 'kinetic energy' and is transported with the 'wave particle' velocity $\mathbf{u} \equiv d\mathbf{x}/dt$. This coincides with the group velocity $\mathbf{u} = \partial\omega/\partial\mathbf{k}$, which for anisotropic but non-dispersive waves, i.e. $c_*(\mathbf{n}) = \omega/k$ with $\mathbf{n} = \mathbf{k}/k$, simplifies to $\mathbf{u} = \partial c_*/\partial\mathbf{n}$. If we consider the energy flux across, say, a large sphere, a directional effect appears, as the component of the energy velocity in the radial direction is $\mathbf{u}.\mathbf{r}$, with $\mathbf{r} \equiv \mathbf{x}/|\mathbf{x}|$. When normalized with the phase velocity, this defines the *directivity factor* $\Lambda \equiv \mathbf{u}.\mathbf{r}/c_*$, which may be calculated from $\Lambda = (\mathbf{r}.\partial/\partial\mathbf{n}) \log c_*$.

The total intensity of radiation, estimated as the energy flux across this large sphere, is $I \sim (\mathbf{E}.\mathbf{r})\,r^2 \sim \rho\Lambda c_*\, v^2 r^2$, and is constant because $v = O(r^{-1})$ from (62). The result for three-dimensional (e.g. slow, fast or acoustic) propagation would become $I \sim \rho c_* v^2 r$ for two-dimensional waves (e.g. cylindrical, as emitted by an infinite line source) because then the velocity field $v = O(r^{-\frac{1}{2}})$. For one-dimensional (e.g. Alfvén) waves there is no decay, and the intensity is just the mean energy flux: $I \sim \bar{E}$. In the case of dissipative propagation the spectrum of the source should be taken as a four-dimensional function in wavenumber, frequency space, so that d^3k is replaced by $d^3k\, d\omega$ in (57), resulting in a wave decay $O(r^{-\frac{3}{2}})$. These and other waves, such as those generated by inflexion edges etc. [cubic and higher-order terms in (59)] in the wavenumber surface (Lighthill 1960), and the non-stationary terms neglected in (61), decay as $v = O(r^a)$ with $a < -1$ (spherically). The intensity of these terms is $I \propto r^2 v^2 \sim O(r^b)$ with $b = 2(1-a) < 0$, and decays to zero at infinity, which means that they do not radiate to the far field.

Thus the only radiating component of the wave field is (62), and the far-field intensity is estimated from $I \sim \rho c_* \Lambda r^2 v^2$ as

$$I \sim \rho(L^2/c_*^5)\, \Lambda U^4\{U^2 + (\mu/4\pi\rho)\, H^2\}^2. \tag{63}$$

In the hydrodynamic case $I \sim \rho(L^2/c_*^5)\, U^8$, so that the radiation is isotropic ($\Lambda = 1$) and scales on the eighth power of velocity, in agreement with aerodynamic acoustics (Lighthill 1952). In the opposite limit when the magnetic pressure predominates (magnetodynamics), $I \sim \rho(L^2/c_*^5)\, \Lambda(\mu/4\pi\rho)^2\, (UH)^4$, so that the radiation is anisotropic ($\Lambda = \Lambda(\mathbf{r})$), involves the ratio of masses $\mu/4\pi\rho$ [see (27)] and scales (symmetrically) on the *fourth* power of velocity and magnetic field, i.e. on $(UH)^4$. In intermediate situations the anisotropy of radiation remains and the intensity still scales on the fourth power of velocity, now multiplied by a bi-quadratic expression in the velocity U^2 and magnetic field H^2 with the ratio of masses $\mu/4\pi\rho$ as coefficient, as stated in the *law of intensity and directivity* of magneto-acoustic radiation (63).

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Appendix. Summary tables

The main results on propagation, generation and radiation have been collected in three summary tables. The material in the tables is included in the account of the corresponding topics in the main text (§§ 2–4). Once the latter has been grasped, the following summary could serve as a short reference on magneto-acoustic waves.

Wave		Propagation		Source
Mode	Condition	Speed	Direction	quadrupole
Magneto-acoustic				
Alfvén	v_3	c_1	$\mathbf{l}$	T_{3i}
Slow	v_a	c_-	$(\mathbf{n}, \mathbf{l})$	T_{ai}
Fast	v_a	c_+	$(\mathbf{n}, \mathbf{l})$	T_{ai}
Limiting waves				
Acoustic	$H = 0$	c_0	$\mathbf{n}$	$T_{ij}^{(0)}$
Alfvén	$\rho' = 0$	c_1	$\mathbf{l}$	$T_{ij}^{(1)} + T_{ij}^{(0)\prime}$
Hydrodynamics				
Fast $\equiv$ acoustic	$\eta \ll 1$	c_0	$\mathbf{n}$	$T_{ij}^{(0)}$
Slow $\equiv$ Alfvén		c_1	$\mathbf{l}$	$\eta^{\frac{1}{2}} T_{ij}^{(0)}$
Hydromagnetics				
Fast $\equiv$ acoustic		c_0	$\mathbf{n}$	$(T_{ij}^{(1)} + T_{ij}^{(0)\prime})/\rho M_0$
Slow $\equiv$ Alfvén	$(c_1/c_0)^2 \ll 1$	c_1	$\mathbf{l}$	$T_{ij}^{(1)} + T_{ij}^{(0)\prime}$
Magnetodynamics				
Fast		c_1	$\mathbf{n}$	
Slow	$\eta \gg 1$	c_0	$\mathbf{l}\}$	$T_{ij}^{(1)}$
Alfvén		c_1	$\mathbf{l}$	

Notes $v_3 \| \mathbf{n} \wedge \mathbf{l}$, $v_a \in (\mathbf{n}, \mathbf{l})$; $\eta \equiv (\mu/4\pi\rho)(H/U)^2$; $M_0 = U/c_0$, $M_1 = U/c_1$
for c_0, c_1, c_+, c_- and $T_{ij}, T_{ij}^{(0)}, T_{ij}^{(0)\prime}, T_{ij}^{(1)}$ (see table 3)

Directions $\mathbf{n} \equiv \mathbf{k}/k$, wave normal; $\mathbf{l} \equiv \mathbf{H}/H$, magnetic; $\mathbf{r} \equiv \mathbf{x}/|\mathbf{x}|$, observation

TABLE 1. Propagation in a range of conditions.

Source generation	R_{ij} Dynamic	S_{ij} Magnetic
Turbulence	$\frac{1}{2}\rho v_i * v_j$	$-(\mu/4\pi) h_i * h_j$
Quadratic stresses	Convection	Magnetic
Order of $\sim$	$\frac{1}{2}\rho v^2$	$\mu h^2/8\pi$
...Pressure	Dynamic	Magnetic
Inhomogeneities	$(p' - c_0^2\rho - \alpha_0^2 s)$	$-(\mu/4\pi)\mathbf{H} * \nabla \wedge (\mathbf{v} \wedge \mathbf{h})$
Zero for homogeneous	Acoustics	Alfvén
Deviation from constant	c_0	c_1

	D_{ij} cause		
Sink	Viscosity	Heat conduction	Electrical resistance
Dissipation			
Diffusion parameter	ν	κ/C_p	$(c^2/4\pi\mu) \times 1/\sigma$
Direct magnitude $\partial/\partial t$	$\partial\tau_{ij}/\partial t$	–	$\dagger H_i * \nabla^2 h_j$
Heat production $\times -\beta_0\delta_{ij}$	$\tau_{k1}\,\partial v_k/\partial x_1\,\partial_{ij}$	$\kappa\nabla^2 T$	$\dagger(\nabla \wedge \mathbf{h})^2$
Dividing term	$1-\beta$	$-\beta$	$1-\beta$
Order of magnitude $\sim$	$\nu U/L$	$(\kappa/C_p)\,U/L$	$\dagger H^2/U^2$
Relative to source	$\nu/\rho UL$	$\kappa/\rho C_p UL$	$c^2/4\pi\mu\sigma UL$
Inverse of	*Re* (Reynolds number)	*Pe* (Péclet number)	*Me* (Magnetic Reynolds number)

Notes $A_i * B_j \equiv A_i B_j + A_j B_i - A_k B_k \delta_{ij}$
$\dagger \equiv c^2/16\pi^2\sigma$; $\beta \equiv (\partial p/\partial l)_T/\rho C_v = R/C_v$

TABLE 2. General and dissipation quadrupoles.

Waves	Acoustic	Alfvén	
Phase speed	$c_0^2 \equiv (\partial p/\partial \rho)_s = \gamma p/\rho$	$c_1^2 \equiv \mu H^2/4\pi\rho = 2P/\rho$	
Direction	$\partial/\partial x_i$	$\partial/\partial l \equiv l_i\, \partial/\partial x_i$	
Modes		Magneto-acoustic	
Phase speed	$c_+ = \frac{1}{2}(\beta^+ + \beta^-)$	$c_- = \frac{1}{2}(\beta^+ - \beta^-)$	c_1
Perturbation	$\in (\mathbf{n}, \mathbf{l})$	$\in (\mathbf{n}, \mathbf{l})$	$\parallel \mathbf{n} \wedge \mathbf{l}$
Designation	*Fast*	*Slow*	Alfvén

Note. $\beta^\pm \equiv |c_0 \mathbf{n} \pm c \mathbf{l}| = \{c_0^2 + c_1^2 \pm 2c_0 c_1 \mathbf{l}.\mathbf{n}\}^{\frac{1}{2}}$

Meaning	*Operator*	Variable
Acoustic	$\partial^2/\partial t^2 - c_0^2\, \partial^2/\partial x_i^2$	$\Delta \equiv \nabla . \mathbf{v}$
Alfvén	$\partial^2/\partial t^2 - c_1^2\, \partial^2/\partial l^2$	$\mathbf{v}_\perp \equiv \mathbf{v} \wedge \mathbf{l}$
	Magneto-acoustic $\square_{ij}\{v_j\}$	
Wave, acoustic	$\square_{ij} \equiv \delta_{ij}\, \partial^2/\partial t^2 - c_0^2\, \partial^2/\partial x_i\, \partial x_j$	
Alfvén	$- c_1^2(\delta_{ij}\, \partial^2/\partial l^2 - l_j\, \partial^2/\partial l\, \partial x_i)$	
Interaction	$+ c_1^2(l_i\, \partial^2/\partial l\, \partial x_j - \partial^2/\partial x_i\, \partial x_j)$	

Quadrupole. $\square_{ij}\{v_j\} = -\rho^{-1}\, \partial^2 T_{ij}/\partial t\, \partial x_j$

$T_{ij} = R_{ij} + S_{ij} - D_{ij}$	Magneto-acoustic
$T_{ij}^{(0)} = \rho v_i v_j + (p - c_0^2 \rho)\, \delta_{ij} - \tau_{ij}$	Hydrodynamic
$T_{ij}^{(1)} = R_{ij} - (c^2/16\pi^2\sigma)\, \nabla^2 h_i * H_j$	Magnetodynamic

Effect	Q	$\mathbf{G}$	W
Cause	Mass source	Applied force	Heat source
Mono/dipole $\times -\rho^{-1}$	$c_0^2\, \partial^2 Q/\partial t\, \partial x$	$\partial G_i/\partial t$	$\beta\, \partial^2 W/\partial t\, \partial x_i$

Note. $T_{ij}^{(0)\prime} = \rho v_i v_j + p\delta_{ij} - \tau_{ij}$; for R_{ij}, S_{ij}, D_{ij} see table 2

Radiation	Directivity	Intensity $\times \rho L^2/c_*^5$
Hydrodynamic	1	U^8
Magnetodynamic	$(\mathbf{r}.\partial/\partial \mathbf{n}) \log c_*$	$(\mu/4\pi\rho)\, U^4 H^4$
Magneto-acoustic	$(\mathbf{r}.\partial/\partial \mathbf{n}) \log c_*$	$\{U^2 + (\mu/4\pi\rho)\, H^2\}^2\, U^4$

TABLE 3. Radiation directivity and magnitude.

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